The exponential function – Part I

Albert A. Bartlett

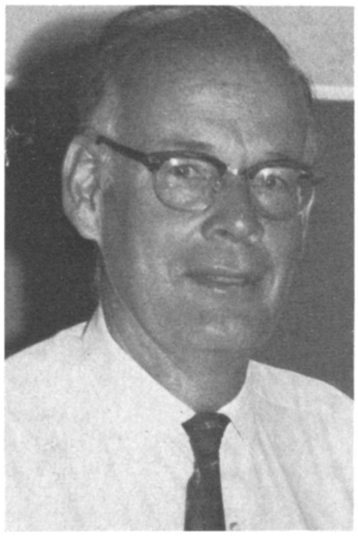
Introduction I.

As the United States celebrates its Bicentennial Year, we can look back at the unparalleled accomplishments of the men and women who have built this country. If any one theme runs through all of our 200-year history, it is *growth*. Our population, our industries, and our consumption of resources have all grown enormously, and are continuing to grow. Our magazines are filled with advertisements that praise our nation's growth.

> "Every school child knows that the Revolution was a struggle for freedom. What is often overlooked is that one of the basic liberties for which the Colonies fought was the freedom of enterprise — the freedom to develop without economic constraints imposed by England.
>
> In the two hundred years of America's growth, freedom of enterprise has been tightly interwoven with our other basic freedoms. It has provided a unique climate for invention, for innovation, and for competition that has allowed our people to achieve an unparalleled living standard. In short, it was and is the most effective, efficient economic system ever devised.
>
> Now, at this special time in our history, Americans should remember that our freedoms are inseparable. Freedom of enterprise is essential to our economic growth and well-being to create more and better jobs, more energy, more security, and the capital that they demand."[1]

While the advertising agencies continue to pour forth unparalleled calls for more growth, a few thoughtful people are beginning to ask, "Can we continue in the future to grow as we have grown in the past?" The answer to this vital question can be found in the mathematics of the exponential function. *The mathematics of growth is the mathematics of the exponential function.*

Growth is one of the great cornerstones of most business and economic systems. The idea that "growth is good" is an item of faith with most Americans. This belief seems to be deeply rooted in the American frontier version of our western European cultural heritage; on the one side, few people ever examine the rational or logical bases for items of faith, and on the other side, I suspect that mathematics and physics teachers who introduce natural logarithms and exponential functions usually illustrate these functions with examples of radioactive or RC *decay*. In introductory physics we seldom mention examples of exponential *growth*, although many such processes dominate our everyday life.



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In this series of articles we will first review the very simple mathematics of the exponential function, and we will cite a number of examples which demonstrate the importance of the function in the real world. These will let us draw some very significant conclusions about our future, about our nation's third century, about our political leaders and about life on "spaceship earth."

If we understand the fundamental arithmetic of growth we will then be able to evaluate many of the conflicting statements that are made almost daily by "experts" and by persons in authority who are reporting on trends or who are making predictions about the future growth of segments of our economy. Few, if any, of these people appear to understand the arithmetic of growth, and unfortunately few Americans have the background that is needed to evaluate these predictions. As a result the predictions go unchallenged. Physics students and teachers have a great responsibility,

1. to understand the problems and perils of growth, and then
2. to alert the public to these problems and perils, even if this means taking issue with the "experts."

The best decisions are those made by an enlightened public. It is our task as students and teachers to help roll away the darkness.

Here is the theme of our presentation:

The greatest shortcoming of the human race
is man's inability to understand
the exponential function.

II. The equation

Exponential growth or decay is a consequence that follows whenever we have a function N which changes with time in such a way that the change ΔN in N during a short time interval Δt is proportional to N and to Δt.

$$\Delta N = kN\Delta t \qquad \text{(II-1)}$$

Rearranging this equation, we get:

$$\frac{\Delta N}{\Delta t} = kN \qquad \text{(II-2)}$$

The time rate of change of the quantity is proportional to the quantity. The larger the value of N the faster it changes.

The meaning of the constant of proportionality k can be seen by rearranging Eq. (II-1)

$$k = \frac{(\Delta N/N)}{\Delta t} \qquad \text{(II-3)}$$

The constant k is then the fractional change ($\Delta N/N$) in N per unit time Δt. The dimensions of k are $(\text{time})^{-1}$ If k had the value "+0.060 per year" it would mean that N was increasing (+) at a rate corresponding to 6% per year. If k had the value " −0.050 per day" it would mean that N was decreasing (−) at a rate corresponding to 5% per day.

III. The function

Equation (II-2) is a differential equation which can be solved in several ways. One way is to ask "What kind of mathematical function has the property that its rate of change is proportional to itself?" When the function N is plotted against the time, t, Eq. (II-2) requires a proportional relationship between the slope ($\Delta N/\Delta t$) and N. Let's examine the graphs and slopes of some simple mathematical functions. (See Fig. III-1.)

If $N = kt$, a linear function, the slope of its graph is a constant: $\Delta N/\Delta t = k$. Evidently the slope is not proportional to N.

If $N = kt^2$, a quadratic function, the slope of its graph is proportional to the first power of time: $\Delta N/\Delta t = 2kt$. Consequently the slope is not proportional to N, which is proportional to t^2.

If $N = k \sin \omega t$, the sinusoidal function, the slope of its graph is proportional to $\cos \omega t$: $\Delta N/\Delta t = \omega k \cos \omega t$. The slope of a sine curve at any point is not proportional to the value of the sine function at that point.

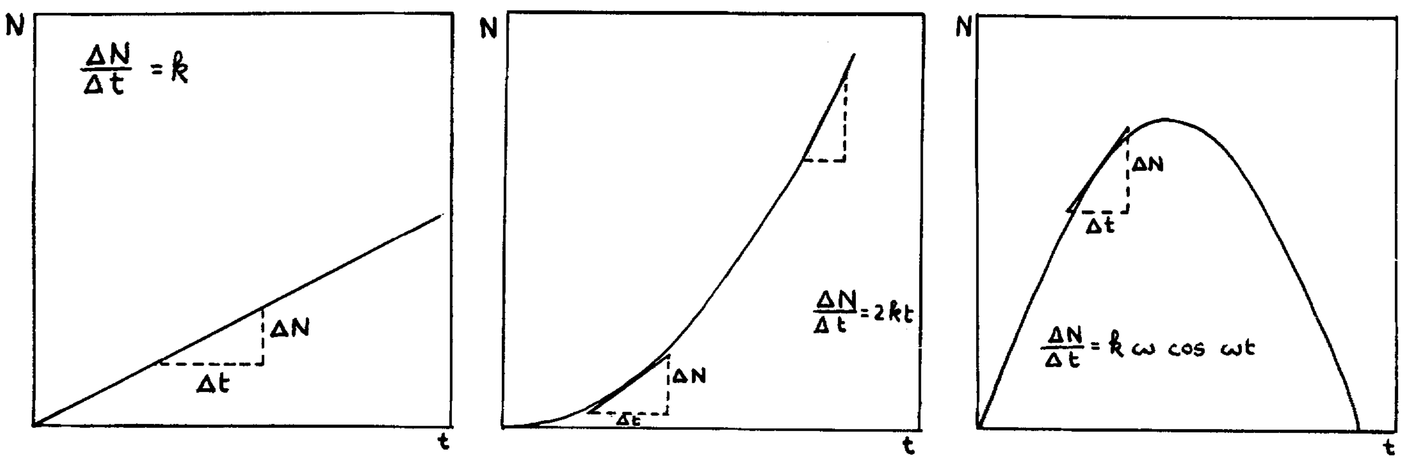
Now let's examine the properties of the function

$$N = N_0 a^{kt} \qquad \text{(III-1)}$$

where a is a constant. Figure III-2 shows a plot of N vs t for $N_0 = 1$, and for $k = 1$ per second. We are thus plotting

$$N = a^t \qquad \text{(III-2)}$$

Fig. III-1. $N = kt$ $\qquad N = kt^2$ $\qquad N = k \sin \omega t$



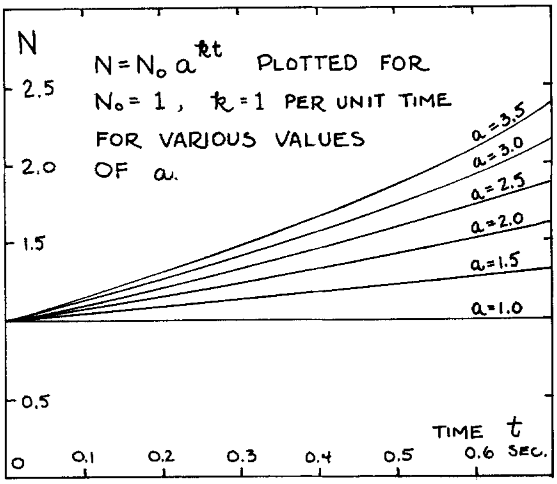


Fig. III-2. The function $N = N_0\, a^{kt}$ is plotted as a function of time for various values of a for the case where $N_0 = 1$ and $k = 1$.

for six different values of a in the range from $a = 1.0$ to $a = 3.5$. For any value of a, the slope of the graph of $N = a^t$ at any point is *proportional* to the value of N at that point. This is the function that we are seeking.

For what value of a is the slope of the function *equal* to the value of the function? [This is required by Eq. (II-2) when k has the value of unity.] In Fig. III-3 is plotted the slope of the curves of Fig. III-2 evaluated at $t = 0$. At $t = 0$ the function N has the value 1.000. . . The slope at $t = 0$ will be equal to the function (that is the slope has the value 1.00 . . .) when a has the value of approximately 2.7. For $a = 2.7$ the slope of the function is proportional to the function, and the constant of proportionality is unity. Thus Eq. (III-1) becomes, approximately,

$$N = N_0\ (2.7)^{kt} \tag{III-3}$$

When evaluated more accurately the number we have called 2.7 has the value 2.71828- - - - -. It occurs so frequently that it is called e and it is used as the base for the system of natural logarithms. The function we seek which obeys Eq. (III-3) is written as

$$N = N_0 e^{kt} \tag{III-4}$$

This function is discussed in calculus terms in Appendix III-1. The lack of understanding of this function may be one of the greatest shortcomings of the human race.

IV. Properties of the function.

A. General properties

Figure IV-1 shows the general properties of the function of Eq. (III-4). The value of the function for $k < 0$ tends toward zero at long times, and the value for $k > 0$ tends to become infinitely large at long times. Between these extremes is the solution for $k = 0$ for which N has the constant value N_0 for all values of t. We will first examine the solutions where $k > 0$.

B. The doubling time

For positive values of k, Eq. (III-4) shows that N

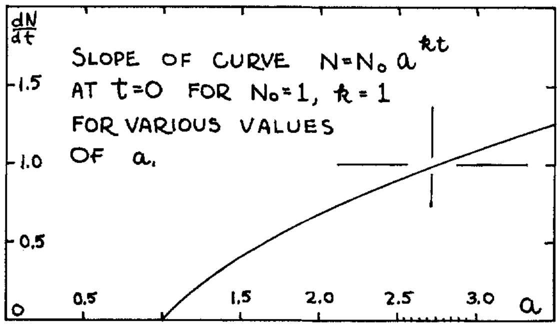


Fig. III-3. The slopes of the curves of Fig. III-2 at $t = 0$ are plotted here as a function of a. The slope has the value equal to the function ($N = 1$, $\Delta N/\Delta t = 1$) at $a = 2.718.\ .\ .\ .$

increases with time, starting with the value $N = N_0$ at time $t = 0$. Let us find the time ($t = T_c$) that is required for N to increase by a factor C from N_0 to CN_0. We use Eq. (III-4).

$$CN_0 = N_0 e^{kT_c}$$

Take the natural logarithm of each side of this equation

$$ln C = kT_c$$

$$T_c = \frac{ln C}{k} \tag{IV-1}$$

One of the most important characteristics of the exponential function is illustrated in Eq. (IV-1). *The time T_c for N to grow by a given factor C remains constant throughout the growth.*

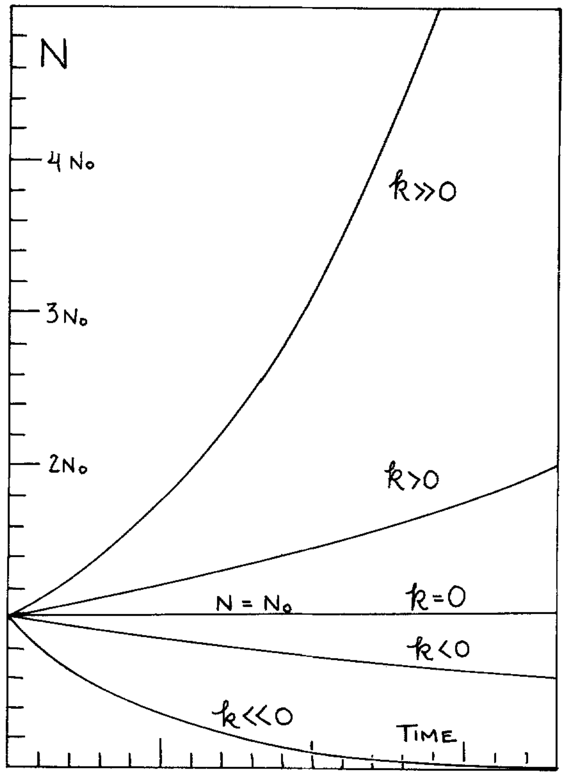


Fig. IV-1. This figure shows a linear plot of the growth or decay of N as a function of time for several different values of the constant k.

If we wish to find the time for N to grow to twice its initial value then we let $C = 2$, and Eq. (IV-1) becomes

$$T_2 = \frac{ln 2}{k} = \frac{0.693...}{k} \quad \text{(IV-2)}$$

C. The growth of the exponential function

The constancy of the doubling time T_2 is a remarkable feature of the exponential function (Eq. III-4). As a result, the growth of N with time has a particularly simple behavior. Starting at N_0 at time $t = 0$, N grows to $2N_0$ at $t = T_2$; it continues to grow to $4N_0$ at $t = 2T_2$; at $3T_2$ it has grown to $8N_0$, and in n doubling times it has grown to $2^n N_0$. This behavior is illustrated in Table IV-1.

Table IV-1

The growth of the exponential function

Time (in units of T_2)	N
$t = 0$	$N = N_0 = 2^0 N_0$
$t = T_2$	$= 2 N_0 = 2^1 N_0$
$t = 2T_2$	$= 4 N_0 = 2^2 N_0$
$t = 3T_2$	$= 8 N_0 = 2^3 N_0$
$t = 10T_2$	$= 1024 N_0 = 2^{10} N_0$
$t = nt_2$	$= 2^n N_0$

D. The doubling times for various growth rates

Let us rewrite Eq. (IV-2)

$$T_2 = \frac{0.693...}{k} = \frac{69.3}{100k} = \frac{69.3}{P} \approx \frac{70}{P} \quad \text{(IV-3)}$$

The quantity P is equal to $100k$ (where k is defined in II-3) and is the instantaneous value of the percent growth rate per unit time. This important relation between the doubling time and the percentage growth rate is illustrated in Table IV-2. This relation may be remembered by observing that the doubling time is given approximately by dividing the number 70 by the percent growth rate. Equation (IV-3) is sometimes given as $T_2 = 72/P$ and Shonle[2] says that this equation, "... is the most useful equation in any question of growth" of which he is aware.

Table IV-2

Doubling times in years for various annual growth rates as calculated from Eq. (I-10).

P (per year)	T_2 (years)
1%	69.3
2%	34.6
3%	23.1
4%	17.3
5%	13.9
6%	11.5
8%	8.66
10%	6.93
15%	4.62
20%	3.46

Time units other than years may be used. Thus if P is the percent growth "per hour," T_2 is the doubling time in "hours."

News stories constantly cite growth rates in percent per year. It is most urgent that we and our students learn to convert growth rates to doubling times by the simple mental arithmetic of Eq. (IV-3).

E. Doing exponential calculations in your head.

One of the most important things we must do in teaching the properties of exponential function is to help students to develop the skill to do exponential calculations in their heads. When a news story about growth makes the observation that the population of the town grew 3% last year, the student should be able to make a quick mental calculation to estimate a) the doubling time for this growth and b) how large a city the town will be if this growth continues for 50 or 100 years in the future. Once a person has developed these skills, he or she can immediately evaluate the impact of a particular growth without having to use a slide rule or an electronic calculator. At this point we can specify three valuable mental skills that are easily developed.

1. *The conversion of growth rates into doubling times.* Growths are almost universally described in terms of percent per unit time but the average reader has no comprehension of the exponential consequences of a given growth rate. Equation (IV-3) is the first weapon in our arsenal of mental arithmetic.

$$\text{Doubling time} = \frac{70}{\text{percent growth per unit time}}$$

Thus when we read that "the tax revenue increased 8% above last year's revenue," we can say that if this continued the revenue would double in about nine years ($70/8 \approx 9$). It would be a great step forward if we could convince the media to report growths in terms of doubling times as well as in terms of percents.

2. *The estimation of long-term overall growth.* In IV C we saw that in n doubling times the quantity N grows from its initial value N_0 to $2^n N_0$. In Appendix III-1 it is shown that this behavior is described by the equation.

$$N = N_0 2^{(t/T_2)} \quad \text{(IV-4)}$$

Suppose we wish to estimate how large N will be after some very large number of doubling times. For instance, let $t = 57\ T_2$. From Eq. (IV-4),

$$N = N_0\ 2^{57}$$

We can estimate the magnitude of 2^{57} in our head by a very simple method. One should remember that $2^{10} = 1024$ so that it is approximately true that

$$2^{10} \approx 10^3 \quad \text{(IV-5)}$$

Thus we may say that

$$\begin{aligned} 2^{57} &= 2^{10}\ 2^{10}\ 2^{10}\ 2^{10}\ 2^{10}\ 2^7 \\ &\approx 10^3\ 10^3\ 10^3\ 10^3\ 10^3 \times 128 \\ &\approx 1.3 \times 10^{17} \end{aligned}$$

This replacement of 2^{10} by 10^3 allows one to estimate overall growth in a very simple way.

Table IV-3

Growth in a lifetime

The right column lists the overall growth (N/N_0) that would result in 69.3 yr for steady growth at the annual percentage rates given in the left column.

Percent annual growth	Growth Factor (N/N_0)
0	$2^0 = 1$
1%	$2^1 = 2$
2%	$2^2 = 4$
3%	$2^3 = 8$
4%	$2^4 = 16$
5%	$2^5 = 32$
6%	$2^6 = 64$
7%	$2^7 = 128$
8%	$2^8 = 256$
9%	$2^9 = 512$
10%	$2^{10} = 1024$

The error in this estimate can also be estimated. Because $2^{10} = 1024$, our replacement of 2^{10} by 10^3 gives a result that is approximately 2.4 percent low. In estimating the value of 2^{57} we made this replacement five times so our estimate (1.3×10^{17}) is low by $5 \times 2.4 = 12$ percent. The actual value of 2^{57} is 1.44×10^{17}.

3. *The estimation of growth in a human lifetime.* The arithmetic of exponential growth becomes particularly simple when it is calculated for a period of 70 years because 70 is approximately 100 ln 2. This is convenient because 70 years is also approximately one human lifetime in the industrially developed countries. For $t = 70$ years, the exponent of 2 in Eq. (IV-4) is $(70/T_2)$ which is $(100k)$ which is the percent annual growth, P (see Eq. IV-3).

Thus in 70 years an annual growth rate of 6% will produce an overall growth of $2^6 = 64$! (See Table IV-3.)

One of our local newspapers recently quizzed members of Boulder's City Council as to what annual rate of growth of Boulder's population each council member felt would be acceptable. The answers ranged from 1% per year to 5% per year. I wrote to the City Council to ask if they realized that if the city's population grew steadily at a modest 5% per year then where we now have one overloaded sewage treatment plant, in one short human lifetime the growth would make it necessary for us to have $2^5 = 32$ overloaded sewage treatment plants.

These three methods of doing mental arithmetic to estimate exponential functions should be made a part of the training of every student we see in our classrooms.

F. Solutions for negative values of k

When k is negative, the quantity N decays from its initial value N_0 toward the ultimate value of zero. Its value decreases to half its initial value in a time called the "half-life" $T_{1/2}$ whose value is the magnitude of the right side of Eq. (IV-2). Thus in one half-life N decays from N_0 to $N_0/2$. In three half-lives, N decays to $N_0/2^3 = N_0/8$, and in n half-lives it decays to $N_0/2^n$. The amount of decay in the first half-life is equal to the sum of all the decays in all the remaining time.

If one starts at $t = 0$ with N_0 atoms of a particular radioactive isotope then the number of atoms remaining undecayed at a time t is given by Eq. (III-4) with $k < 0$. The magnitude of k is the fraction of the atoms that decay per unit time, and the half-life $T_{1/2}$ is

$$\frac{(ln\ 2)}{|k|}$$

Different radioactive species have different half-lives and these can range from less than 10^{-6} sec to greater than 10^9 yr.

When a capacitor C is given a charge Q_0 and then has a resistor R connected across its two plates at time $t = 0$ the charge Q remaining on one plate at any later time t is given by

$$Q = Q_0 e^{-t/RC}$$

where in this case, the k of our standard formula is equal to $-1/RC$.

The same mathematical forms are encountered in describing the absorption of X rays or gamma rays in varying thickness of absorbers.

G. The semi-logarithmic plot

The rapid growth of N (for $k > 0$) and the rapid decay (for $k < 0$) are difficult to plot on linear graphs. To get around these graphical difficulties it is common to plot the logarithm of N instead of N itself, as a function of time. If we take the natural logarithm of both sides of Eq. (III-4)

$$ln\ N = ln\ N_0 + kt$$

This is the equation of a straight line ($ln\ N$ vs t) where $ln\ N_0$ is the intercept and k is the slope. Such a plot is called a semi-logarithmic plot and it is most easily made on "semi-log" graph paper where the marked distances on the ordinate scale are proportional to the logarithms of numbers instead of being proportional to the numbers themselves. The slopes of the straight lines are positive for $k > 0$ and are negative for $k < 0$. Figure IV-2 shows a semi-log plot of growing exponentials (curves A and B have $k > 0$) and a decaying exponential (curve C has $k < 0$).

H. The determination of constants from semi-log plots

It is very easy to determine the value of the contant k from a semi-log plot. For the case of positive k (Fig. IV-2, curve A) it is noted that N increases from 5 to 10 in a time of 8.5 h. Therefore 8.5 is the doubling time, and the constant k is $0.693\ldots/8.5 = 0.0816$ per hour. On curve B we can see ten doublings from $N = 0.50$ at $t = 0$ to $N = 512$ at $t = 13.0$. Therefore the doubling time for this curve is $(13.0)/10 = 1.30$ h, and the constant k is $(0.693\ldots/1.30) = 0.533$ per hour. In curve C we see a decrease from the value of 160 to the value 10 in 8 h. This is a decay by a factor of 2^4 or through 4 half-lives. The half-life is $8/4 = 2.0$ h and $k = -0.693\ldots/2.0 = -0.346$ per hour.

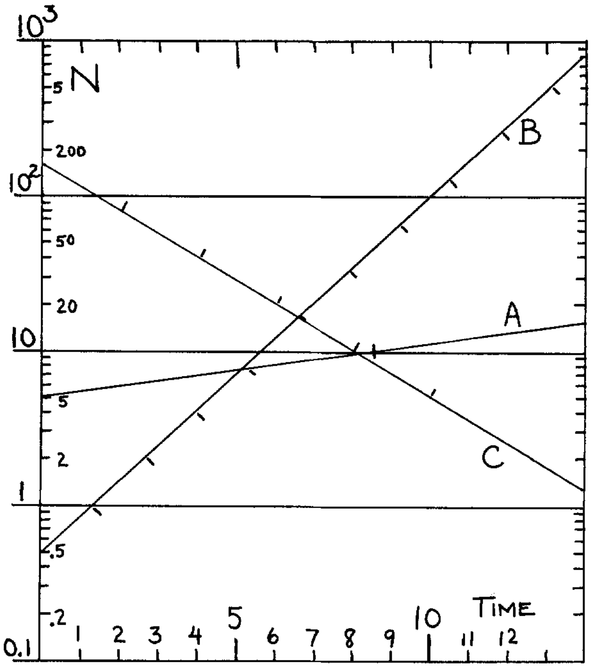


Fig. IV-2. Semi-logarithmic plots of growing exponentials (curves A and B) and of a decaying exponential (curve C). The intervals between marks on curves A and B are doubling times, and between marks on curve C are half-lives. The units of time are hours.

I. An interesting detail

If $k = +0.060$ per year, what is the growth in one year? Before you answer, "six percent," think of Eq. (III-4)

$$N = N_0 e^{0.060 \times 1} = 1.06184 N_0$$

Thus a value of k = "+0.060 per year" produces an actual growth of 6.184% in one year!

If a quantity that is growing exponentially actually increases by 6.00% in one year, what is the value of k?

$$1.0600\, N_0 = N_0 e^{k \times 1}$$

$$k = \ln 1.0600 = +0.05827$$

The reason for the apparent discrepancy between the value of k and the growth in one year is illustrated in Fig. IV-3. The upper curve is a plot of the exponential function $N = N_0 e^{0.060t}$ while the lower curve is the straight line $N = N_0\ (1 + 0.060t)$ which is tangent to the exponential curve at $t = 0$. The slope of the straight line is constant so that it describes an increase in N from N_0 to $1.060N_0$ in the time interval from $t = 0$ to $t = 1$. From Eq. (II-2) the slope $\Delta N/\Delta t$ of the exponential curve is kN. Thus the slope of the exponential curve must increase as N increases. This is responsible for the rise of the exponential curve above the straight line even though both curves have the same slope at $t = 0$. The quantity k is not the change in N in a finite interval of time but is the *instantaneous value* of the fractional rate of change of N.

In many instances, such as those involving population growth, the data are not sufficiently precise to allow a meaningful differentiation between the two values (k and the actual annual fractional growth) and in these cases, if a population grows 6% per year we will describe its growth by saying $k = +0.060$.

V. The power of powers of two

The story of the exponential function can best be told as the story of powers of two. The rapidity with which powers of two can grow to overwhelming numbers is fascinating, frightening, and largely unappreciated by the average person. Let us illustrate by some examples.

A. The king and the mathematician. The story is told of the court mathematician who invented the game of chess for his patron the king. The game so pleased the king that he offered to reward the mathematician for the invention. The mathematician said, "Give me one grain of wheat (2^0) on the first square of the chessboard, two grains (2^1) on the second square, four grains (2^2) on the third square, eight grains (2^3) on the fourth square, etc., doubling the number

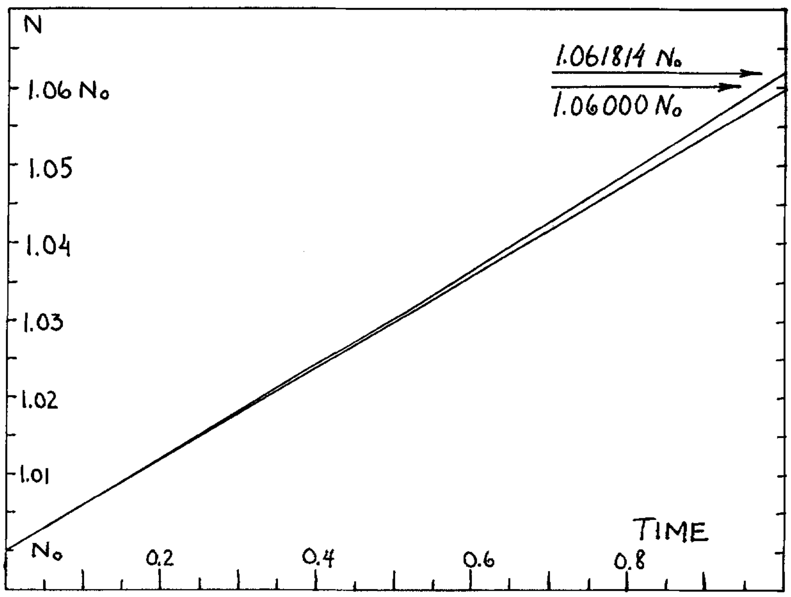


Fig. IV-3. The upper line is the exponential curve for $k = +0.06$ and the lower line is the straight line $N = N_0\ (1 + 0.060\,t)$. The straight line is tangent to the curve at time $t = 0$.

Table V-I

Wheat on the Chessboard
(The payment of a modest debt)

Number of the square	Grains of wheat on the square	Total grains of wheat on the board thus far
1	$1 = 2^0$	$1 = 2^1 - 1$
2	$2 = 2^1$	$3 = 2^2 - 1$
3	$4 = 2^2$	$7 = 2^3 - 1$
4	$8 = 2^3$	$15 = 2^4 - 1$
5	$16 = 2^4$	$31 = 2^5 - 1$
6	$32 = 2^5$	$63 = 2^6 - 1$
n	2^{n-1}	$2^n - 1$
64	2^{63}	$2^{64} - 1$

of grains on each succeeding square until all the squares have been used." The king thought that the mathematician was foolish to ask for such an apparently trivial reward. But the king changed his mind when he started counting out the grains of wheat. Table V-1 shows the number of grains required on each square. The plan calls for 2^{63} grains to be placed on the 64th square! When this has been done, there will be $(2^{64} - 1)$ grains of wheat "on the chessboard." Let us convert 2^{64} to powers of 10 by the approximate methods of Sect. IV, E,2.

$$2^{64} = 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 2^4$$
$$= 16 \times 10^{18}$$

(The actual value is 18.45×10^{18})

Since common grains of wheat have a mass of 3.4 g per hundred, the total mass of wheat needed to repay the debt is 6.27×10^{14} kg which is approximately 500 times the current annual world-wide harvest of wheat. *This apparently modest method of rewarding the mathematician for inventing the game of chess would require an amount of wheat which is probably larger than the total amount of wheat that has been harvested in the entire history of the earth!*

Another very important aspect of repeated doublings is seen in Table V-1. *The number of grains placed on any square is one grain larger than the total of all of the grains on all of the preceeding squares.*

B. Folding paper in half.
The sheets of paper on which *The Physics Teacher* is printed are 5.7×10^{-3} cm. If one folded such a sheet of paper in half, the thickness of the two halves would add to $2 \times 5.7 \times 10^{-3}$ cm. If one folds the paper in half again the four layers of paper will have a total thickness of $4 \times 5.7 \times 10^{-3}$ cm. When the sheet has been folded five times it will have the thickness of $(2^5 = 32)$ sheets which is the thickness of the normal issue of *The Physics Teacher* minus the front and back covers. (The normal issue has 64 numbered pages on 32 turning pages.) How thick would the sheet from *The Physics Teacher* be when it has been folded a mere fifty times? The answer is both simple and amazing.

$$T = 2^{50} \times 5.7 \times 10^{-3}\,\text{cm}$$
$$= 6.4 \times 10^{12}\,\text{cm}$$
$$= 6.4 \times 10^{7}\,\text{km}$$

This is 0.43 of the distance from the earth to the sun!
With these two examples we can begin to appreciate the awesome power of powers of two.

C. The growth of electric power generating capacity in the United States. The electric generating capacity in the United States has doubled approximately every ten years for a century! Advertisements suggest that it is expected that the U.S. generating capacity will double again in the next 10 to 12 years. We can draw several conclusions from this.

1. The ten-year doubling time corresponds to an annual growth rate of 7% [see Eq. (IV-3)].
2. The growth of the number of kilowatts of electrical generating capacity in the United States during the past century can be approximated by the equation

$$KW = Kw_0 e^{0.070t} = KW_0 2^{t/10}$$

   where t is measured in years.
3. In the next doubling time of 10 to 12 years the U.S. will consume more electrical energy than has been consumed heretofore in the entire previous history of the electric generating industry in the United States (Remember the grains of wheat on the chessboard, and Table V-1.).
4. If this 7% annual growth were to continue for the next 70 years the electrical generating capacity of the U.S. would grow by a factor of $2^7 = 128$ and if it were to continue until the nation's tricentennial, the generating capacity would have to grow by a factor of $2^{10} = 1024$! If all of this power is to be generated by the use of fossil fuels, then where we have one coal mine today we would have to have 1024 coal mines century from now. If all of this power is to be generated in steam plants (fossil fuels or nuclear) having the same thermodynamic efficiency as today's generating plants, then for every megawatt of heat that is dumped into the environment today there will be a thousand megawatts dumped into the environment one century from now!

D. Observations. These three examples permit us to make three important points about exponential growth, and these lead us to a compelling conclusion.

1. Repeated doubling leads to astronomical numbers.
2. Even modest annual growths can, in a relatively few years, lead to incredible overall growth.
3. An understanding of the exponential function has a great importance in the real world.

The compelling conclusion to which we are led is that nothing can grow exponentially for very long. In particular, *the current large annual consumption of resources cannot grow through many more doubling times.*

In spite of this obvious and simple conclusion, we find that "experts" would have us believe that the current energy problem is merely a minor transient on a curve of steady growth of the rate of consumption of energy to which there will be no end. Consider these words which are taken from a recent full-page full-color advertisement in a national news magazine![5] (The nonsentences are in the original.)

> "America depends on electricity. Our need for electricity actually doubles about every 10 or 12 years. Can we keep meeting this need year after year? It depends on what we do in the next few years. We have the technology to make all the electricity we need... And now some of our fuels are in short supply. We are going to have to rely more and more on our resources that are in ample supply. Coal and nuclear fuel for example. (sic) We are going to have to build more nuclear power plants. And new, more efficient coal-burning plants, (sic). *But we have to use this electricity and all our resources wisely* (emphasis is mine). Because America depends too much on electricity to ever run out." (sic)

The closing line is a Madison Avenue masterpiece. It establishes in the reader's mind the curious and illogical conclusion:

Because America depends so much on electricity, we will never run out!

With this example of the growth of the electric power industry in the United States, we suddenly see that the exponential function has enormous importance in the real world. With this example, our discussion makes the transition from a review of abstract arithmetic to the discussion of the exponential problems of the world in which we live. We can begin to see that simple growth can have alarming consequences. We can begin to suspect that the "energy crisis" was predictable, and we can see that no glib solutions can cover up the grim and fascinating realities of the exponential function in our everyday life.

As physics teachers we usually introduce our students to the exponential function when we discuss the *RC* circuit where

$$\frac{dQ}{dt} = -\left(\frac{1}{RC}\right) Q$$

or when we discuss radioactive decay, where

$$\frac{dN}{dt} = -kN$$

I suspect that it is rare for a physics teacher to go on from these introductions and say whenever we have

$$\frac{dN}{dt} = kN, \qquad \text{(III-1)}$$

we have exponential growth. I suspect that few teachers ever show their students how many important economic and social phenomena are described (at least approximately) by Eq. (II-2) and which consequently exhibit exponential growth. I suspect that few teachers of physics point out to their students the many important applications of exponential growth in the real world, where its importance is so great that it can be said that

The greatest shortcoming of the human race
is man's inability to understand
the exponential function.

In succeeding issues of *The Physics Teacher* we will examine the role of the exponential function in the world in which we live.

Acknowledgments

The series of articles, of which this is the first, grew out of a lecture on the arithmetic of growth which I prepared and first gave in 1969 to students in a class on problems of the environment. Since that time it has been given with evolving variations more than two dozen times to classes and to civic and service groups. I wish to thank colleagues and friends for their many contributions to the ideas I am putting forth here. Thanks also go to the Physics Faculty of Colgate University for their kindness in allowing me to use an office during the summer of 1975 when much of the manuscript was completed. I am particularly indebted to the secretarial staff of the Department of Physics and Astrophysics and of the Joint Institute for Laboratory Astrophysics (both in Boulder) for typing the many drafts of the manuscript.

Appendix III-1

Equation II-2 may be written

$$\frac{dN}{dt} = kN. \qquad \text{(1a)}$$

The solution is of the form

$$N = N_0\, a^{bt} \qquad \text{(2a)}$$

where a and b are constants. The time derivative of N is

$$\frac{dN}{dt} = b\,(\ln a)\, N_0\, a^{bt} = b\,(\ln a)\, N \qquad \text{(3a)}$$

This satisfies Eq. (1a) provided

$$k = b \ln a. \qquad \text{(4a)}$$

We must choose values of a and b that satisfy Eq. (4a) where k is the instantaneous value of the fractional change in N per unit time.

If we choose to let $a = e$ where

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 2.718 \ldots\ldots \quad (5a)$$

then $k = b \ln e = b$ and the solution to Eq. (1a) is

$$N = N_0 e^{kt} \quad (6a)$$

It is convenient to let $a = 2$ in which case $b = k/\ln 2$. This is recognized as the reciprocal of the doubling time

$$b = \frac{1}{T_2}$$

and then the solution of Eq. (1a) is

$$N = N_0\, 2^{t/T_2} \quad (7a)$$

It is also convenient to let $a = 10$ in which case $b = k/\ln 10 = k/2.303 = 0.4343k$. Now the solution of Eq. (1a) is

$$N = N_0\, 10^{0.4343\, kt} \quad (8a)$$

Part II of this article will appear in a future issue.

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